KLEIN LINK MULTIPLICITY AND RECURSION

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ABSTRACT. The \((m, n)\)-Klein links are formed by altering the rectangular representation of an \((m, n)\)-torus knot. Using the braid representation of a \((m, n)\)-Klein link, we generalize a previous braid word result and show that the \((m, 2m)\)-Klein link can be expressed recursively. Applying braid permutations, we demonstrate that Klein links have a formulaic multiplicity and classify the Klein links that are equivalent to knots.

1. Introduction

Introduced in [3] and reinterpreted in [4], Klein links are the result of altering the rectangular diagram for an \((m, n)\)-torus knot. Instead of the standard edge orientations for a torus, the orientations are modified to match the rectangular diagram for a Klein bottle as in Figure 1. Upon identification of the edges, the strings form a link on the Klein bottle which is called a Klein link [3, 4]. A necessary caveat to this creation procedure is forcing the Klein bottle into 3-dimensions since all links are trivial in 4-dimensions [4]. To maintain the knottedness of the links, we consider a particular punctured Klein bottle where the puncture occurs in the top-left corner of the rectangular diagram, as demonstrated in Figure 1 [4]. Changing the location of the puncture corresponds to changing crossings in the resulting link [4]. To avoid this, we assume that all strands on the vertical edges will be below this puncture and all strands on the horizontal edges will remain above the puncture [3, 4]. While this is a strict requirement, it leads to nice results regarding the links that are formed by this process.

Given that the immediate relationship between Klein links and torus knots, it is not surprising that Klein links would similarly have “nice” formulas for various link invariants. Building from previous results, we will demonstrate that Klein links similarly have a predictable multiplicity and that certain “larger” Klein links are related to smaller Klein links. While some basic similarities are shown in [3] and [4], including the equivalence of the \((m, 2)\)-Klein link and

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Figure 1. The formation of the \((3, 2)\)-Klein link [4].

\((m-1, 2)\)-torus knot, it has been shown that Klein links have a predictable braid representation [4].

Given the significance of braid representations to our discussion of Klein links, we properly introduce braids and essential notation from [4] in Section 2. Then we consider the natural homomorphism from the braid group to the permutation group in Section 3. In Section 4, we use the braid form of a Klein link to demonstrate a recursive construction of certain Klein links, a generalization of a result in [4], and determine the multiplicity of an arbitrary Klein link. Finally, we discuss implications of our results and future work in Section 5.

2. Braid Representations

The **braid group** on \(n\) strings, \(B_n\), is a group under composition, with identity 1, generated by \(\sigma_1, \sigma_2, \ldots, \sigma_{n-1}\) which satisfy the following **braid relations** [6, 7]:

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2
\]

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, 2, \ldots, n - 2
\]

The elements \(B \in B_n\) are called **braids** on \(n\) strings, where the **braid word** of \(B\) is the sequence of generators composed to form \(B\). Two braids, \(B, B' \in B_n\) are **braid equivalent**, which we will denote by \(B = B'\), if \(B\) and \(B'\) can be related by a finite sequence of braid relations [1].
The braid group also has a geometric interpretation, where a braid is represented by $n$ disjoint interwoven strings, labeled 1 through $n$, with distinct endpoints [6]. In this setting, each $\sigma_i$ corresponds to a crossing involving strings $i$ and $i+1$ [6]. Moreover, each braid relation, in addition to the group axiom of inverses, corresponds to an isotopy of the braid that fixes the ends of each string. Using the standardization of [1], the generator $\sigma_i$ corresponds to the $i$th string crossing over the $(i+1)$st string as shown in Figure 2. Consequently, $\sigma_i^{-1}$ corresponds to the $i$th string crossing under the $(i+1)$st string.

By identifying the strings at one end of the braid to the corresponding strings at the other end, as in Figure 3, we obtain the **closed braid** of a braid [1].

Using the closure operation, every braid corresponds to some link. Moreover, Alexander’s Theorem states that every link can be represented as a closed braid [1, 6]. Since we are interested in closed braids, we also consider a weaker form of equivalence. Two braids $B_1$ and $B_2$ are **Markov equivalent**, denoted $B_1 \leftrightarrow B_2$, if the closed braids are ambient isotopic [1, 7]. Under Markov equivalence, we allow two additional relations, called **Markov moves**, on a braid $B \in \mathcal{B}_n$:

**Conjugation:** For $1 \leq i \leq n-1$, $B \leftrightarrow \sigma_i B \sigma_i^{-1} \leftrightarrow \sigma_i^{-1} B \sigma_i$.

**Stabilization:** $B \leftrightarrow B \sigma_n \leftrightarrow B \sigma_n^{-1}$. 

![Figure 2. Braid group generator $\sigma_i$ and inverse [4].](image1)

![Figure 3. Geometric braid with braid word $\sigma_1^{-2}\sigma_3^{-1}\sigma_2\sigma_1\sigma_3^{-1}$ and its closure [4].](image2)
Markov’s Theorem states that any two Markov equivalent braids can be related by a finite sequence of braid relations and Markov moves [1, 6, 7]. Consequently, it is sufficient to consider any braid form of a link.

Before discussing the braid word for a Klein link, we have one final piece of notation to introduce. For \( k \geq l \geq 1 \), we write \( \Gamma_{l,k} = \prod_{i=l}^{k} \sigma_i = \sigma_l \sigma_{l+1} \cdots \sigma_k \) [4]. It naturally follows that \( (\Gamma_{l,k})^{-1} = \sigma_k^{-1} \sigma_{k-1}^{-1} \cdots \sigma_l^{-1} \) [4]. In this way, we simply write the braid for the \((m,n)\)-torus knot \( T(m,n) = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^m = (\Gamma_{1,n-1})^m \).

Derived in [4], the \((m,n)\)-Klein link has a braid word that corresponds to \( T(m,n) \) composed with the half-twist on \( n \) strings. Thus the \((m,n)\)-Klein link has a braid word on \( n \) strings of the form

\[
K(m,n) = (\Gamma_{1,n-1})^m \prod_{i=1}^{n-1} (\Gamma_{i,n-1})^{-1}.
\]

3. Permutations of Braids

We now turn our discussion toward the symmetric group on \( n \) elements, denoted \( S_n \), which consists of permutations of elements from a finite set of size \( n \), \( X_n \). In accordance with the notation of [5], the permutation \((1 5)(2 3)\) represents the bijection given by \( 1 \mapsto 5, 5 \mapsto 1, 2 \mapsto 3, 3 \mapsto 2, 4 \mapsto 4 \). Furthermore, we express the composition of permutations right-to-left.

Given a braid, we can determine how the strings interact in the closed braid by considering the permutation of strings at the end of the braid. There is a unique group homomorphism \( \pi : B_n \to S_n \) given by \( \pi(\sigma_i) = (i \ i + 1) \) [6]. For a braid word \( B \), \( \pi(B) \) is known as the braid permutation [7]. In the case of \((m,n)\)-torus knots, the relevant braid permutation is the \( m \)-fold composition of

\[
T_n = (1 \ n \ n-1 \ \cdots \ 2).
\]

Since a half-twist switches the order of strings in a braid, the corresponding braid permutation must switch the order of elements. So, as the braid of an \((m,n)\)-Klein link is the composition of \( T(m,n) \) with a half-twist, the braid permutation of an \((m,n)\)-Klein link is \( T_n^m \) composed with the product of \( \left\lceil \frac{n}{2} \right\rceil \) disjoint transpositions on the right:

\[
K_n = (1 \ n)(2 \ n-1) \cdots \left( \left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n+1}{2} \right\rceil \right)
\]
Note that $(K_n)^2 = (1)$ since disjoint transpositions commute and have order 2. Geometrically, if we were to apply the half-twist twice, we would produce a full twist in the braid and so every string would return to its original position [6].

One way to explicitly express $T_n$ and $K_n$ makes use of the integers modulo $n$: $T_n(i) \equiv i - 1 \mod n$ and $K_n(i) \equiv -i + 1 \mod n$. The advantage with this form of these permutations is that it is easier to examine compositions than using cycle notation.

Our use for the braid permutation depends on the use of group actions. For a group $G$ acting on a set $X$, we denote the fix of $g \in G$ by $fix(g) = \{ x \in X \mid g \cdot x = x \}$ [5]. The orbit of $x \in X$ is $orb_G(x) = \{ g \cdot x \mid g \in G \}$ [5]. In general, the orbits of a braid permutation correspond to the formation of a knot in the closure of the braid. Thus counting the number of orbits, which we denote $|X/G|$, in a braid permutation is equivalent to determining the multiplicity of the link. Burnside’s Lemma states that [5]

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |fix(g)|.$$

Using subgroups of $S_n$ to act on a braid with $n$ strings, we will show that Klein links have a predictable multiplicity. For the remainder of our discussion, we assume that we use the trivial group action and so permutations maps to themselves identically. The underlying set for this action will be the strings of the braid, using the labels 1 through $n$, denoted $[n] = \{1,2,\ldots,n\}$.

4. Results

In [4], we showed that $K(n,n) = \prod_{i=1}^{n-1} \Gamma_{1,n-i}$ under braid equivalence. Since $K(n,n) = \Gamma_{1,n-1}K(n-1,n)$, we can trivially extend this result for $(n-1,n)$-Klein links. That is, for $n > 1$,

$$K(n-1,n) = \prod_{i=1}^{n-2} \Gamma_{1,n-1-i}.$$

Using this extension of the theorem, we can further generalize this result for a larger class of Klein links.

**Theorem 4.1.** If $m \geq n - 1 > 0$ then $K(m,n) = (\Gamma_{1,n-1})^{m-n+1} \prod_{i=1}^{n-2} (\Gamma_{1,n-1-i})$. 
Proof. Using the simplified braid word for $K(n - 1, n)$,

$$K(m, n) = (\Gamma_{1,n-1})^m \prod_{i=1}^{n-1} (\Gamma_{i,n-1})^{-1}$$

$$= (\Gamma_{1,n-1})^{m-(n-1)} \left( (\Gamma_{1,n-1})^{n-1} \prod_{i=1}^{n-1} (\Gamma_{i,n-1})^{-1} \right)$$

$$= (\Gamma_{1,n-1})^{m-n+1} K(n - 1, n)$$

$$= (\Gamma_{1,n-1})^{m-n+1} \prod_{i=1}^{n-2} (\Gamma_{1,n-1-i}).$$

Hence $K(m, n) = (\Gamma_{1,n-1})^{m-n+1} \prod_{i=1}^{n-2} (\Gamma_{1,n-1-i})$ as desired. □

The following lemma will allow us to demonstrate a recursion relationship on a certain subset of the $(m, n)$-Klein links.

Lemma 4.2. For $n - 1 > m \geq 0$, $K(m, n) = \prod_{i=1}^{m-1} (\Gamma_{1,m-i}) \times \prod_{i=m+1}^{n-1} (\Gamma_{i,n-1})^{-1}.$

Proof. We will first consider two special cases: $m = 0$ and $m = 1$. When $m = 0$, $K(0, n) = \prod_{i=1}^{n-1} (\Gamma_{i,n-1})^{-1}$. For $m = 1$,

$$K(1, n) = \Gamma_{1,n-1} \prod_{i=1}^{n-1} (\Gamma_{i,n-1})^{-1} = \prod_{i=2}^{n-1} (\Gamma_{i,n-1})^{-1}. $$

Now assume that $m > 1$. Since $m < n - 1$ and $\Gamma_{1,n-1}(\Gamma_{1,n-1})^{-1} = 1$,

$$K(m, n) = (\Gamma_{1,n-1})^m \prod_{i=1}^{n-1} (\Gamma_{i,n-1})^{-1} = \left( (\Gamma_{1,n-1})^m \prod_{i=1}^{m} (\Gamma_{i,n-1})^{-1} \right) \prod_{i=m+1}^{n-1} (\Gamma_{i,n-1})^{-1}. $$

It is now sufficient to show that $(\Gamma_{1,n-1})^m \prod_{i=1}^{m} (\Gamma_{i,n-1}^{-1}) = \prod_{i=1}^{m-1} (\Gamma_{1,m-i})$ under braid equivalence. This is easily demonstrated using a finite induction technique from [4]. For the first step,

$$(\Gamma_{1,n-1})^m \prod_{i=1}^{m} (\Gamma_{i,n-1}^{-1}) = (\Gamma_{1,n-1})^{m-1} \prod_{i=2}^{m} (\Gamma_{i,n-1}^{-1}).$$
For the second step, take \( B_2 = \prod_{i=3}^{m-1} (\Gamma_{i,n-1})^{-1} \). Then

\[
(\Gamma_{1,n-1})^{m-1} \prod_{i=2}^{m} (\Gamma_{i,n-1})^{-1} = (\Gamma_{1,n-1})^{m-2} (\Gamma_{1,n-1}(\Gamma_{2,n-1})^{-1}B_2) = (\Gamma_{1,n-1})^{m-2} (\Gamma_{1,1}B_2).
\]

Since \( B_2 \) consists of generators acting on strings labeled 3 or greater, \( B_2 \) and \( \Gamma_{1,1} \) use disjoint subsets of strings and so \( \Gamma_{1,1}B_2 = B_2\Gamma_{1,1} \). Hence

\[
(\Gamma_{1,n-1})^{m-1} \prod_{i=2}^{m} (\Gamma_{i,n-1})^{-1} = (\Gamma_{1,n-1})^{m-2} \left( \prod_{i=3}^{m} (\Gamma_{i,n-1})^{-1} \right) \Gamma_{1,1}.
\]

In the \( k \)th step, set \( B_k = \prod_{i=k+1}^{m} (\Gamma_{i,n-1})^{-1} \). Then

\[
\Gamma_{1,n-1}(\Gamma_{k,n-1})^{-1}B_k = \Gamma_{1,k-1}B_k = B_k\Gamma_{1,k-1}.
\]

Hence in the \( k \)th step, the given braid has a braid word of the form \( (\Gamma_{1,n-1})^{m-k} B_k \prod_{i=1}^{k-1} (\Gamma_{1,k-i}) \).

In the \( m \)th step, \( B_m = 1 \) and so the braid is given by \( \prod_{i=1}^{m-1} (\Gamma_{1,m-i}) \) as desired. \( \square \)

We can apply our two previous results to obtain the following theorem.

**Theorem 4.3.** The \((m,2m)\)-Klein link is the disjoint union of the \((m,m)\)-Klein link and the \((0,m)\)-Klein link.

**Proof.** We will first consider the special case where \( m = 1 \). Since

\[
K(1,2) = (\Gamma_{1,1})^1 \prod_{i=1}^{1} (\Gamma_{1,i})^{-1} = 1,
\]

the \((1,2)\)-Klein link is the closure of the trivial braid on 2-strings and hence the disjoint union of two unknots. Furthermore, \( K(1,1) \) and \( K(0,1) \) are trivial braids on one string and so the \((1,1)\)- and \((0,1)\)-Klein links are equivalent to the unknot. Thus the \((1,2)\)-Klein link is the disjoint union of the \((1,1)\)-Klein link and the \((0,1)\)-Klein link.

Now we assume that \( m > 1 \). By Lemma 4.2,

\[
K(m,2m) = \prod_{i=1}^{m-1} (\Gamma_{1,m-i})^1 \prod_{i=m+1}^{2m-1} (\Gamma_{i,2m-1})^{-1}.
\]
Let $B_1 = \prod_{i=1}^{m-1} (\Gamma_{1,m-i})^{-1}$ and $B_2 = \prod_{i=m+1}^{2m-1} (\Gamma_{i,2m-1})^{-1}$. Then $B_1$ and $B_2$ act on disjoint sets of strings and so their closures are disjoint. Moreover, $K(m,m) = B_1$ by Theorem 4.1.

Consider $K(0,m) = \prod_{i=1}^{m-1} (\Gamma_{i,m-1})^{-1}$ as a braid on $2m$ strings. Applying the bijection $f(i) = i + m$ to the labeling of the generators for the $(0,m)$-Klein link, we obtain

$$K(0,m) = \prod_{i=1}^{m-1} (\Gamma_{i,m-1})^{-1} = \prod_{i=m+1}^{2m-1} (\Gamma_{i,2m-1})^{-1} = B_2.$$

Hence the $(m,2m)$-Klein link is the disjoint union of the $(m,m)$-Klein link and the $(0,m)$-Klein link. \hfill \Box

We have only talked about specific subsets of Klein links to this point. However, some results, such as the number of components in a given Klein link, apply to all Klein links. To this end, we first examine the $m$-fold composition of $T_n$.

**Lemma 4.4.** For all $n, m \in \mathbb{N}$ and $i \in [n]$, $T_n^m(i) \equiv i - m \mod n$.

**Proof.** We proceed by the First Principle of Mathematical Induction on $m$. In the case $m = 1$, $T_n^1(i) \equiv i - 1 \mod n$ by definition. Now assume the result holds for some $k < m$ and consider $T_n^{k+1}(i)$:

$$T_n^{k+1}(i) \equiv T_n(T_n^k(i)) \equiv T_n(i - k) \equiv (i - k) - 1 \equiv i - (k + 1) \mod n.$$

Hence $T_n^m(i) \equiv i - m \mod n$. \hfill \Box

Our second lemma shows that a certain subgroup of $S_n$ has predictable order. The proof makes use of the fact that a congruence, $ax \equiv b \mod n$, has gcd($a,n$) solutions if and only if $b \mid n \ [2]$.

**Lemma 4.5.** Let $\langle T_n^m K_n \rangle$ be the subgroup of $S_n$ generated by $T_n^m K_n$. If $n > 2$ then $|\langle T_n^m K_n \rangle| = 2$.

**Proof.** First we show that $T_n^m K_n \neq (1)$. To the contrary, suppose $T_n^m K_n = (1)$. Then using associativity of composition and $(K_n)^2 = (1)$,

$$(T_n^m K_n)K_n = (1)K_n \Rightarrow T_n^m = K_n.$$
By Lemma 4.4, $T_m(i) \equiv i - m \mod n$ and $K_n(i) \equiv -i + 1 \mod n$. Since $T_n^m = K_n$,

$$i - m \equiv -i + 1 \mod n \Rightarrow 2i \equiv m + 1 \mod n.$$ 

The number of solutions to this congruence depends on $\gcd(2, n)$. Since $n > 2$, $\gcd(2, n) = 1$ if $n$ is odd and $\gcd(2, n) = 2$ if $n$ is even. Thus there are, at most, two solutions to the congruence. Hence at most two strings could have been mapped to the same position by $T_m^n$ and $K_n$. However, as both permutations are acting on a braid with at least 3 strings, $T_m^n \neq K_n$. Hence $T_m^n K_n \neq (1)$.

Now we show that $(T_m^n K_n)^2 = (1)$. Let $i \in [n]$ and consider $K_n T_m^n K_n(i)$:

$$K_n T_m^n K_n(i) \equiv K_n T_m^n (-i + 1) \equiv K_n ((-i + 1) - m) \equiv -(i + 1 - m) + 1 \equiv i + m \mod n.$$ 

Then it follows that

$$(T_m^n K_n)^2(i) \equiv T_m^n (K_n T_m^n K_n)(i) \equiv T_m^n (i + m) \equiv (i + m) - m \equiv i \mod n.$$ 

Thus $(T_m^n K_n)^2$ maps every string to itself identically and so $(T_m^n K_n)^2 = (1)$ as desired. Hence $\langle T_m^n K_n \rangle = \{(1), T_m^n K_n\}$ and so $|\langle T_m^n K_n \rangle| = 2$. □

While this results requires a braid with more than two strings, $n > 2$, we can also consider the special cases where $n = 1, 2$. If $n = 1$, $T_m^1 K_1 = (1)$ independent of our choice of $m$ and so $|\langle T_m^1 K_1 \rangle| = 1$. When $n = 2$, $T_m^2 K_2 = (1 2)^{m+1}$. If $m$ is odd, $(1 2)^{m+1} = (1 2)$ and so $|\langle T_m^2 K_2 \rangle| = 1$. However, if $m$ is even, $(1 2)^{m+1} = (1 2)$. In this case, $\langle T_m^2 K_2 \rangle = \{(1), (1 2)\}$ and $|\langle T_m^2 K_2 \rangle| = 2$.

We apply Lemma 4.5 and these special cases to prove our final result.

**Theorem 4.6.** Let $m, n \in \mathbb{N}$. If $m$ is even then the $(m, n)$-Klein link has multiplicity $\left\lceil \frac{n}{2} \right\rceil$. If $m$ is odd then the $(m, n)$-Klein link has multiplicity $\left\lceil \frac{n+1}{2} \right\rceil$.

**Proof.** First we consider the cases for $n = 1, 2$. If $n = 1$, then $|\text{fix}(1)| = 1$, $|\langle T_m^1 K_1 \rangle| = 1$ and so, by Burnside’s Lemma,

$$\left\lceil \frac{1}{1} \right\rceil = 1.$$ 


Now assume that $n = 2$. Then $|\text{fix}(1)| = 2$. If $m$ is odd, $|\langle T^m_2 K_2 \rangle| = 1$ and, by Burnside’s Lemma,

$$\frac{|[2]/\langle T^m_2 K_2 \rangle|}{1(2)} = \frac{1}{2}.$$  

If $m$ is even, $|\langle T^m_2 K_2 \rangle| = 2$. Since $T^m_2 K_2 = (1 2)$ fixes no elements of $[2]$, $|\text{fix}(T^m_2 K_2)| = 0$. Applying Burnside’s Lemma,

$$\frac{|[2]/\langle T^m_2 K_2 \rangle|}{2(2)} = 1.$$  

Now assume that $n > 2$. By Lemma 4.5, $|\langle T^m_n K_n \rangle| = 2$. Since $(1)$ fixes the set $[n]$, $|\text{fix}(1)| = n$. To count the number of orbits of $T^m_n K_n$ acting on $[n]$ using Burnside’s Lemma, we also require $|\text{fix}(T^m_n K_n)|$. By definition, we want to find $i \in [n]$ such that $T^m_n K_n(i) = i$. Using Lemma 4.4,

$$T^m_n K_n(i) \equiv T^m_n (−i + 1) \equiv −i − m + 1 \mod n.$$  

So we consider

$$−i − m + 1 \equiv i \mod n \Rightarrow 2i \equiv −m + 1 \mod n.$$  

To solve this congruence, we examine three distinct cases:

**Case 1:** Suppose $n$ is odd. Then $\gcd(2, n) = 1$ and so $2i \equiv −m + 1 \mod n$ has one solution. Hence $|\text{fix}(T^m_n K_n)| = 1$. By Burnside’s Lemma,

$$\frac{|[n]/\langle T^m_n K_n \rangle|}{1} = \frac{n + 1}{2}.$$  

**Case 2:** Suppose $n$ and $m$ are even. Then $\gcd(2, n) = 2$. Since $m$ is even, $−m + 1$ is odd and thus $\gcd(2, n) = 2 \nmid (−m + 1)$. Hence $2i \equiv −m + 1 \mod n$ has no solutions. Thus $|\text{fix}(T^m_n K_n)| = 0$. By Burnside’s Lemma,

$$\frac{|[n]/\langle T^m_n K_n \rangle|}{2} = \frac{n + 2}{2}.$$  

**Case 3:** Suppose $n$ is even and $m$ is odd. As before, $\gcd(2, n) = 2$. Since $m$ is odd, $−m + 1$ is even and so $2i \equiv −m + 1 \mod n$ has two solutions. It follows that $|\text{fix}(T^m_n K_n)| = 2$. Hence, applying Burnside’s Lemma,

$$\frac{|[n]/\langle T^m_n K_n \rangle|}{2} = \frac{n + 2}{2} = \left\lceil \frac{n}{2} \right\rceil.$$  

Hence the number of orbits, and thus multiplicity of an $(m, n)$-Klein link, is as desired.  \(\square\)
As an immediate consequence of this result, we are able to determine the values of $m$ and $n$ for which a $(m, n)$-Klein link is a knot.

**Corollary 4.7.** A $(m, n)$-Klein link is a knot if and only if either $n = 2$ and $m$ is even or $n = 1$.

It has previously been shown that the $(m, 1)$-Klein link is the unknot for $m \geq 1$ and the $(m, 2)$-Klein links are equivalent to the $(m - 1, 2)$-torus knots [3, 4]. Thus every knot produced by the Klein link construction is a torus knot.

5. Conclusions

Using the braid representation of a Klein link, we have shown that certain Klein links are recursively built from other Klein links. It is possible that other types of Klein links show a similar pattern, but such a pattern has not yet been found in other Klein links.

Applying braid permutations, we proved that the multiplicity of an $(m, n)$-Klein link depends only on $n$ and the parity of $m$. Our formulaic expression is likely a result of the relationship between Klein links and torus knots. This is supported by the fact that every Klein link which is equivalent to a knot is a torus knot. Further research may find additional similarities, especially using invariants such as braid index or crossing number. Other areas of potential interest include finding a formula for the linking number, given that we are able to predict when two components of a Klein link interact, and determining the $n$-colorability of a Klein link.

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References


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