A Preliminary Study of Klein Knots
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ABSTRACT
Knots are used in everyday life, from tying shoelaces to untangling extension cords. Knots also have significance in the mathematical world, with applications in fields such as biology and physics. Considerable work has been done categorizing knots and an important category that has arisen is the class of torus knots. We begin work that analyzes knots on the non-orientable counterpart of the torus – the Klein bottle.

1. INTRODUCTION
A major topic within the realm of knot theory is the classification of different types of knots. Since the development of knot theory, a variety of different classes of knots such as virtual knots, satellite knots, torus knots, and hyperbolic knots have been examined. Knot theory includes the study of properties of knots in these classes.

A brief discussion of terminology will provide ample background for those unfamiliar with knot theory. A link is a subset of Euclidean space that consists of $m$ disjoint, piecewise linear, simple closed curves. Each of the $m$ simple closed curves within a link is a component. We say a link having one component is a knot [7]. We can think of a knot as one piece of string that has been knotted and the ends of the string subsequently “glued” together. This fits the typical perception of a knot with some differences – the knot has no thickness and friction is irrelevant. The simplest of all knots is a string that forms a circle in space; this is known as the unknot. It is also worth noting that every knot is a link, but the converse is not true.

Though knots reside in three-dimensional space, they are most commonly studied through projecting them onto a two-dimensional plane. When we do this projection, note that there can be points where the string crosses itself. Conveniently, this is known as a crossing.

Figure 1: Two-dimensional projection of a knot

Every knot projection has a given number of crossings. Among all possible projections of a knot, the fewest number of such crossings for a given knot is called the crossing number of the knot. Corresponding to manipulations of a knot in three-dimensions, we have two-dimensional deformations on knot projections called planar isotopies. As a result of Kurt Reidemeister's work (1927), these planar
isotopies boil down to three moves that may be performed on the projected knot. These are known as Reidemeister moves [7].

Figure 2: The three Reidemeister moves

One very productive approach to knot theory research is to identify particular types or classes of knots and use the characteristics of the class to obtain results for that knot type. Torus and satellite knots are two such classes.

2. TORUS KNOTS

Torus knots are one of the most studied classes of knots. A torus knot is constructed by wrapping a string around a torus and, without allowing the string to cross itself, gluing the ends of the string together. The resulting knot is known as a \((p,q)\)-torus knot, where \(p\) and \(q\) are integers. The value of \(p\) denotes the number of times the string intersects the longitude curve while \(q\) is the number of times that the string passes through the meridian curve. Note that the longitude curve is a loop around the hole of the torus while the meridian curve is a loop through the hole of a torus.

A useful way to visualize torus knots is on a rectangular representation of the torus [9]. We obtain this representation by first cutting along a meridian curve and deforming the torus to obtain a cylinder (Figure 4a).

Now, cutting along a longitude curve and laying the cylinder flat, we arrive at the rectangular form of a torus. The torus can now be viewed as this rectangle with the directional identifications given by arrows as in Figure 4b.

Let us now look at what happens to a torus knot in this transformation from a torus to a rectangle. See Figures 5a and 5b for the transformation of a \((1,2)\)-torus knot.
3. SATELLITE KNOTS

A second important class of knots is the class of satellite knots. Satellite knots, together with torus knots and a third class of knots (hyperbolic knots) form a particular partition of the set of all knots [1]. A satellite knot is formed in three steps. First, a knot is encased on the inside of a solid torus (Figure 6). Next, we cut the torus on a meridian curve. Finally, the torus is knotted as though it was a string and the ends are reconnected. The net result is the equivalent of a knot within a knot. The knotting of the torus is just an embedding of the torus into the 3-sphere, $S^3$ [7]. This “surgery” procedure described here is the process that inspired our exploration into Klein knots and is also at the center of a potentially related line of research in Dehn Surgery [6]. Dehn Surgery on knots has been used in the study of 3-manifolds. It has been shown that all 3-manifolds can be obtained by using this technique on knot complement spaces. Current work on the classification of 3-manifolds explores the creation of Klein bottles within the Dehn surgery process [3, 4].

4. KLEIN KNOTS

Given the wealth of interesting and useful results found in the study of torus knots, it is natural to explore knots on the torus’ non-orientable partner, the Klein bottle. The Klein bottle is best described by using a rectangular representation with identifications similar to that used for a torus [9]. We take the rectangle from Figure 4b and alter the top identification direction as shown in Figure 7a.

The identification of the vertical sides produces a cylinder as with the torus construction (Figure 7b). However, for the Klein bottle, the end circles on the cylinder have opposite orientation. So instead of attaching them in the natural way as was done to form a torus, we must take one end of the cylinder inside the cylinder and attach the circles. This process can only be done in four-dimensional space but there are a number of three-dimensional representations used to visualize the Klein bottle. The most common of these is shown in Figure 8.
The lower part of the cylinder bends upward and passes through a hole in the upper part of the cylinder in order to attach the top and bottom circles with the correct identification direction. In an actual Klein bottle, this hole does not exist since the fourth dimension is used to obtain the proper identification.

In this initial exploration, we simply consider knots and links derived from torus knots when the torus is cut and reattached to form the three-dimensional representation of the Klein bottle shown in Figure 8. We call these Klein knots. Since torus knots can be displayed clearly on the rectangular representation of the torus, we use this representation to produce the Klein knot associated to each of the torus knots. Figures 9a through 9d below show the process, beginning with the rectangular presentation of a (1,2)-torus knot and ending with the corresponding Klein knot. As Figure 9d shows, a (1,2)-Klein knot is an unlink of two components. This is distinct from the (1,2)-torus knot which is an unknot.

We note that there have to be some standardization decisions made in the method used for constructing this particular representation of the Klein bottle.

For clarity, we create the cylinder from the rectangle by first identifying the vertical (left and right) sides. In doing so, we position the cylinder so that the left half of the rectangle becomes the front of the cylinder. The identified sides form the left vertical line on the cylinder and the center vertical of the rectangle forms the right vertical line of the cylinder. Consistency in this process is important as even a rigid rotation of the cylinder can create problems with proper strand re-attachment along the identification circles.

Second, it is important that the placement of the hole in the cylinder be done with care as it can lead to inconsistencies in the reattaching of the knot strands. If the hole is moved
around the rectangle or cylinder, one must not change its position in relation to any of the strands on the rectangle or cylinder. Doing so amounts to changing crossings on the final Klein bottle and thus will change the knot. For the results in this paper, we place the hole in the region determined by the knot strands that includes the upper left corner of the rectangle representation (Figure 10).

![Figure 10: A (1,2) torus knot with Klein bottle hole](image)

It is useful to think of this process as a manifold surgery similar to that done to create the class of satellite knots from a torus. In this case, however, we have knots on the surface of the torus rather than knots encased by a torus. Our process cuts a torus and reattaches it in the three-dimensional representation of the Klein bottle. The torus knot simply follows the surface through the process. We are interested in studying what knots result when applying this process to various torus knots. We seek patterns in this relationship and in the types of knots obtained from certain classes of torus knots. We would also like to determine whether or not the transformed knot could be obtained from its corresponding torus knot in some consistent way through Reidemeister moves and crossing changes.

5. RESULTS
As stated earlier, (p,q)-torus knots are equivalent to (q,p)-torus knots. In the Klein knot case, a similar property is not true. For instance, a (2,1)-Klein knot is an unknot (see Theorem 2 below); whereas a (1,2)-Klein knot is a disjoint union of two unknots (Figure 9d). Further, it can be shown that a (p,0)-torus knot is the unlink of p components. It is worthy to note (in the form of the following theorem) that the Klein knot case is identical.

**Theorem 1:** A (p,0)-Klein knot is the unlink of p components.

*Proof:* Given a (p,0)-Klein knot, we begin by laying out p strands on the rectangular representation of the Klein bottle.

![Figure 11: The p strands on the rectangular representation of a Klein bottle](image)

Then identifying the vertical sides, starting below the hole, we obtain a cylinder with p disjoint rings.

![Figure 12: After the first identification of p strands](image)

Once the cylinder is glued with the Klein orientation, we will still have p ‘rings’ or unknots surrounding the resulting Klein bottle.
Although we have established that \((p,q)\)-Klein knots and \((p,q)\)-torus knots may result in different knots, another similar pattern emerges between the \((p,1)\)-Klein knots and \((p,1)\)-torus knots. In particular, both are the unknot, a known result for the \((p,1)\)-torus knot [8].

**Theorem 2:** A \((p,1)\)-Klein knot is the unknot.

**Proof:** Lay out \(p+1\) strands on the rectangular representation of a Klein bottle with the hole in the standard position.

![Figure 14: The \(p+1\) strands on the rectangular representation of a Klein knot](image1)

Note that \(p\) strands will be located below the hole, while one strand will be in the northeast corner of the diagram. When identifying the vertical sides at step two, we obtain at a helix on the cylinder, having \(p-1\) complete revolutions.

![Figure 15: After the first identification of \(p+1\) strands](image2)

In steps three and four, the \(p-1\) complete turns are preserved and the northernmost arc is glued to the southernmost arc (Figure 16). This will create a strand on the front of the resulting Klein knot (Figure 17).

![Figure 16: The final stages for a \((p,1)\)-Klein knot](image3)

Using Type II Reidemeister moves, we can move this front strand over and to the outside of the \(p-1\) complete turns of the helix. After a series of \(p-1\) Type I Reidemeister moves, we can untwist the \(p-1\) revolutions. The result is the unknot.

![Figure 17: The resultant \((p,1)\)-Klein knot](image4)
With the exception of the (1,2)-Klein knot, the results we have presented thus far for Klein knots are identical to those for torus knots. Also, all are the unknot or unlink of \( p \) components. One example where a Klein knot is distinct from the corresponding torus knot and both are nontrivial is the (3,2) case. The simplest non-trivial example of a torus knot is a (3,2)-torus knot; it is the trefoil [1]. We determine that a (3,2)-Klein knot is a Hopf link (Figure 19).

A further promising area is found in cable knots. A cable knot is a satellite knot in which the initial knot (the one encased in a solid torus) is a torus knot [7]. Thus, our Klein knots are closely related to cable knots. Instead of surgery that reattaches the cut torus exactly as it was cut, we are reattaching in a non-orientable way. Our approach is the beginnings of a possible extension of cable knots into non-orientable spaces. The Dehn surgery work in [3,4,5] is applied to non-cabled knots.

6. AREAS FOR FURTHER STUDY
Our categorization of Klein knots is far from complete. We are interested in developing analogous results for Klein knots that are known for torus knots – seeking additional patterns in the groupings of Klein knots. Additional areas for research include:

- classifying Klein knots with an alternative placement of the hole
- determine when the result is a non-trivial knot
- analyzing knots on other three-dimensional representations for a Klein bottle and on the actual Klein bottle in four dimensions
- examination of the actual crossing changes that occur in the torus knot to Klein knot changing process.

7. REFERENCES


