The Classification of a Subset of Klein Links

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ABSTRACT
This paper focuses on Klein links, a non-orientable counterpart to the commonly studied class of torus links. In particular, a formula for determining the crossing numbers of specific Klein links is derived. Further, a reduction algorithm is developed that decomposes links contained within a particular subset of Klein links into two disjoint links. This reduction will simplify future discussion of Klein links.

1. INTRODUCTION
Knot theory, a relatively new field of mathematics, explores the defining characteristics of knots and links. A link is a subset of three-dimensional Euclidean space that consists of \( m \) disjoint, piecewise linear, simple closed curves, where each simple closed curve within a link is a component \([8, 5]\). A link with only one component is a knot; thus, every knot is a link, but most links are not knots \([8, 5]\). A mathematical link can be thought of as a finite number of strings (each representing a component) tangled or knotted in any manner, and then the two ends of each distinct string segment are connected. Figure 1 is an example of a picture or projection of a link.

Every knot and link is classified based on its inherent qualities. Torus knots are an extensively studied class with many categorized knots and known relationships. Torus knots represent the specific class of knots that lie on the surface of a torus without any intersections \([4]\). This paper focuses on Klein links, which are closely related to torus knots. Similarly, Klein links lie on the surface of a three-dimensional, once punctured, Klein bottle without any intersections. In particular, the knot invariant crossing number for a specific subset of Klein links will be analyzed.

\( \text{Definition 1. The crossing number of a knot or link } A \text{, denoted } c(A), \text{ is the fewest number of crossings that occurs in any projection of the knot or link } [4]. \)

The location where a component of a link crosses another component or itself is called a crossing.
Two crossings are seen in Figure 1.

The crossing number of a link can be found through the use of Reidemeister moves. Reidemeister moves change a projection of a link by manipulating crossings; however, the resulting link is equivalent \[4\]. These three moves are illustrated in Figure 2.

![Reidemeister Moves](image)

**Figure 2: Reidemeister moves \[1\].**

**Definition 2.** The mirror image of a knot or link \(A\) denoted \((A)^*\), is a knot or link, in which each crossing is switched so that the segment of the link that crossed over the other section of the link, now crosses under it.

![Mirror Images](image)

**Figure 3: \(4_2\) link on the left and its mirror image \(4_2^*\) link on the right.**

The naming convention and classification of links will correspond to that of Rolfsen, as presented in \[9\]. As our specific set of Klein links has been researched, the resulting links have been catalogued based on invariants. The results presented here will extend a large portion of the catalogue through both explicit and reductive techniques.

2. **TORUS LINKS**

As previously noted, a link that can be placed on the surface of a torus in a way such that it does not cross over itself is called a torus link. Torus links are denoted \(T(m, n)\) because they can be classified by the number of times the string wraps around the longitude (\(m\) times) and the meridian (\(n\) times) (see Figure 4).

![Torus Surface](image)

**Figure 4: Torus surface \[5\].**

An alternative method for creating a torus link is by first placing strands on the oriented rectangular representation of a torus surface as demonstrated in Figure 5. Corresponding to \(T(m, n)\), there are \(m\) strands down the left side of the rectangular diagram with identification, while \(n\) strands are across the top. The vertical edges of the rectangle connect to form a cylinder. Then, the horizontal edges connect to create the three-dimensional torus.

![Torus Knot](image)

**Figure 5: \(T(3, 2)\) trefoil knot \[2, 3\].**

3. **KLEIN LINKS**

Similarly, Klein links may be formed using an identified rectangular diagram for a once punctured Klein bottle as seen in Figure 6. As with torus links, there are \(m\) strands down the left side of the rectangular diagram, while \(n\) strands are across the top. To form our specific set of Klein links,
the “hole” in the upper left hand corner represents the self intersection of the three-dimensional Klein bottle. Every Klein link investigated in this work is developed using the same representation of the Klein bottle to maintain consistency. The vertical edges of the rectangle are connected to form a cylinder. Then, the horizontal edges are subsequently connected by deforming the bottom of the cylinder to pass through itself and connect with the top of the cylinder as seen in Figure 6 and Figure 7. The particular Klein links analyzed here can be classified based on their invariants after they are removed from the once punctured Klein bottle.

4. BRAIDS

A different approach to analyzing links uses braids, as every link can be represented as a braid [4]. Using this alternate method, links are analyzed in a new manner, which can be beneficial in the classification and categorization of links. A braid is composed of a number of strings, which are all connected to both a “imaginary” top and bottom bar [4]. As the strings cross over and under each other, the braid as a whole progresses downward. The closure of a braid results from the connection of the top and bottom bar, creating a corresponding link. Consider a braid with \( n \) strings numbered 1, 2, \ldots, \( n \) from left to right. Using this labeling convention, the braid can be algebraically described using braid generators as seen in Figure 8. These braid generators represent over or under crossings, where \( \sigma_i \) represents string \( i \) crossing over string \( i+1 \) and \( \sigma_i^{-1} \) denotes string \( i+1 \) crossing over string \( i \).

Analogous to the Reidemeister moves, braid relations can be used to manipulate a braid into an equivalent braid [4]:

- Rule 1: For all \( i \), \( \sigma_i \sigma_i^{-1} = 1 = \sigma_i^{-1} \sigma_i \)
- Rule 2: For all \( i \), \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \)
- Rule 3: For \( |i - j| > 1 \), \( \sigma_i \sigma_j = \sigma_j \sigma_i \).

Once a braid has been written in terms of \( \sigma_i \) and \( \sigma_i^{-1} \), these rules can then be used to simplify the braid. This simplification process can yield important results about invariants regarding a link that corresponds to a particular braid.

As previously discussed, torus links have been studied extensively and, as a result, a general form for their braid word has been developed.

**Proposition 1.** A general braid word for a torus knot or link \( T(m, n) \) is \((\sigma_1 \sigma_2 \ldots \sigma_{n-1})^m\) [4].

This proposition for torus links is also very useful in the study of Klein links. Using the torus link braid word as a reference point, it can be determined that the general braid word for a Klein link is the product of the torus link braid word with a half twist to the left.
Proposition 2. The formation of a $K(m,n)$ Klein link composes the braid word of the $T(m,n)$ with the half twist, $\prod_{i=1}^{n-1}(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_{i}^{-1})$ (see Figure 9) resulting in the following general braid word for a $K(m,n)$:

$$K(m,n) = (\sigma_1\sigma_2\cdots\sigma_{n-1})^m \prod_{i=1}^{n-1}(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_{i}^{-1}).$$

Figure 9: Half twist on an $n$ string braid; equivalent to the $K(0,n)$. With this information, braid words for certain Klein links can be generated. In particular, the subset $K(0,n)$ will be explored.

Corollary 1. A general braid word for a $K(0,n)$ is

$$\prod_{i=1}^{n-1}(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1}\cdots\sigma_{i}^{-1}).$$

Proof. Examine $K(0,n)$, where $m = 0$. The resulting $n$-strand braid word is equivalent to the half twist from Definition 2 and the closure of the resulting braid is the $K(0,n)$ (Figure 9).

Since the $K(0,n)$ is equivalent to only a half twist on an $n$ string braid, the number of crossings in the general braid word follows directly from the formula of the half-twist (Corollary 1).

Corollary 2. A general braid word for the $K(0,n)$ has

$$\left(\sum_{i=1}^{n-1} i \right) = \frac{n(n-1)}{2}$$

crossings.

Figure 10: General braid diagram for even (left) and odd (right) values of $n$.

Determining the number of components of a Klein link is one invariant for distinguishing one link from another. If two links have different numbers of components, they cannot possibly be equivalent; thus, calculating the number of components in a link can be useful in their classification and categorization. Here, the $K(0,n)$ is again considered for two different cases of $n$, when $n$ is even and when $n$ is odd.

Lemma 1. If $n$ is even, then the $K(0,n)$ has $\frac{n}{2}$ components. If $n$ is odd, then $K(0,n)$ has $\frac{n+1}{2}$ components.

Proof. For all $K(0,n)$, the first strand and the $n^{th}$ strand form one component of the resulting link, the second strand and the $n-1^{st}$ strand form the second component and so forth (see Figure 10). In this manner, it is easily concluded that an even $n$ results in $\frac{n}{2}$ components. Similarly for an odd $n$, the middle strand will always form one component (when the braid is closed), which results in $\frac{n+1}{2}$ components. See Figure 10 for the general braid diagrams for $K(0,n)$ (even and odd $n$).

5. CROSSING NUMBER

The following result examines two conditions for $n$ to determine the crossing number of $K(0,n)$.

Theorem 1. For $n$ even, $n \geq 2$:

$$c(K(0,n)) = \frac{n(n-2)}{2}.$$
Figure 11: Closure of the general $K(0, n)$ braid with $n$ even (The dashed component illustrates the behavior that each additional component would exhibit).

Proof. As seen in Figure 11, each component of a $K(0, n)$ Klein Link (with $n$ satisfying the conditions for this case), shares 4 crossings with each other component, and has one crossing with itself. This is true for all $K(0, n)$ (even $n$), when the general braid is closed. A Type I Reidemeister move may be used to eliminate the left most crossing in this in Figure 11. Similarly, a series of Type III and Type I Reidemeister moves can be used to eliminate each other crossing of a components and itself. This reduced diagram of a $K(0, n)$, where each component’s intersection with itself is removed is shown in Figure 12.

Figure 12: Reduction of a $K(0, 6)$; the dashed component’s behavior illustrates the general behavior of additional components.

It is important to note that any two components in a $K(0, n)$, where $n$ is even, form a $4_2^{*}$ link (as seen in Figure 3). Since the $4_2^{*}$ has a crossing number of 4, and all components in the diagram (see Figure 12) are in this minimal crossing state, the number of crossings in the link cannot be reduced further. Thus the crossing number for these links can be obtained by taking the sum of the crossings of our general braid word (Corollary 2) and subtracting one crossing for each component (Lemma 1). The trivial case when $n$ is equal to 2 and an unknot is formed is included in this result. Thus for $n$ even, $n \geq 2$:

$$c(K(0, n)) = \frac{n(n-1)}{2} - \frac{n}{2} = \frac{n(n-2)}{2}.$$  

**Theorem 2.** For $n$ odd, $n \geq 3$:

$$c(K(0, n)) = \frac{(n-1)^2}{2}.$$  

**Proof.** This case with $n$ odd follows similarly from Theorem 1.

Figure 13: Closure of the general $K(0, n)$ braid where $n$ is odd (The dashed component illustrates the behavior that each additional component would exhibit).

When comparing the unsimplified $K(0, n)$ link resulting from these two cases, the only change is an addition of the center component in the odd case (see Figure 11 and Figure 13). This component has two crossings with every other component, but never crosses itself. Using the same sequence of Reidemeister moves described in the previous proof, all of the crossings in which a component crosses itself, may be eliminated.

If any two components are examined except the
center component, a $4_2^*$ link is formed (equivalently to the even case). If we now look at the innermost component with any other component, a Hopf link ($2_1^1$) is formed (as seen in Figure 1). As in the even case, since neither a $4_2^*$ link nor a $2_1^1$ link can be simplified to have fewer crossings, this diagram represents a projection with the fewest number of crossings. Compare this to the previous diagram illustrates the general behavior of additional components.

Figure 14: Reduction of a $K(0.7)$; the dashed component’s behavior illustrates the general behavior of additional components.

By combining Theorems 1 and 2, for a $K(0. n)$ where $m < n$ will be investigated in order to determine relations among these Klein links. To condense the repetitive nature of the braid words corresponding to Klein links, the following notation is introduced.

**Definition 3.** If $l \leq k$ let

$$\Gamma_{l,k} = \sigma_l \sigma_{l+1} \sigma_{l+2} \ldots \sigma_{k-1} \sigma_k$$

(in ascending order) and if $l \geq k$ let

$$\Gamma_{l,k} = \sigma_l \sigma_{l-1} \sigma_{l-2} \ldots \sigma_{k+1} \sigma_k$$

(in descending order).

**Definition 4.** Let

$$\Gamma_{r,s}^{-1} = \sigma_r^{-1} \sigma_{r+1}^{-1} \sigma_{r+2}^{-1} \ldots \sigma_{s-1}^{-1} \sigma_s^{-1}$$

if $r \leq s$ and let

$$\Gamma_{r,s}^{-1} = \sigma_r^{-1} \sigma_{r-1}^{-1} \sigma_{r-2}^{-1} \ldots \sigma_{s+1}^{-1} \sigma_s^{-1}$$

if $r \geq s$.

Using this notation, the braid relations imply the following rules:

- **Rule 1:** For all $i$ and $j$,
  $$\Gamma_{i,j} \Gamma_{j,i} = 1 = \Gamma_{j,i} \Gamma_{i,j}$$

- **Rule 2:** For all $|j - i| = 1$,
  $$\Gamma_{i,j} \Gamma_{j,i} = \Gamma_{j,i} \Gamma_{i,j}$$

- **Rule 3:** For $a < b, c < d$, and $c - b > 1$,
  $$\Gamma_{a,b} \Gamma_{c,d} = \Gamma_{c,d} \Gamma_{a,b}.$$

**Note 1.** A $K(q, q)$ braid representation is equivalent to its reflection over a vertical axis.

Consider the general braid word for $K(q, q)$, which can be seen in Proposition 2. Once braid moves have been applied, the $K(q, q)$ can be written as:

$$(\sigma_1 \sigma_2 \sigma_3 \ldots \sigma_{q-2} \sigma_{q-1})(\sigma_1 \sigma_2 \sigma_3 \ldots \sigma_{q-3} \sigma_{q-2}) \ldots$$

If this braid were to undergo a reflection over a vertical axis, the left side of the braid would then become the right side of the new reflected braid and the right side of the original braid would become...
the left side of the new reflected braid. As nothing about the link has changed except its location, a vertical reflection of a braid merely changes the projection of the link not the actual link it represents. Strands that were overstrands would now be understrands in the reflected braid. Strands that were understrands in the initial braid would then be overstrands in the reflected braid. The strand that was the first strand would then be the last strand and the strand that had been last in the initial braid would be first in the reflected braid.

This concept is illustrated in Figure 15 where the first strand became the third and the third strand became the first. Notice that, through braid moves, it can be shown that these two braids are equivalent, meaning they are just different projections of the same link. The superscripts on the \( \sigma_i \) terms do not change. This further shows that the two braids are equivalent, just different projections of the same link in braid form. Thus, the braid word of the reflected braid for the reduced \( K(q,q) \) is given by

\[
(\sigma_{q-1}\sigma_q-2\sigma_q-3\ldots\sigma_2\sigma_1) \\
(\sigma_{q-1}\sigma_q-2\sigma_q-3\ldots\sigma_3\sigma_2)\ldots(\sigma_{q-1}\sigma_q-2)\sigma_{q-1},
\]

where the first strand is now the last strand, the second strand is now the second to last strand and so on.

The previous definitions are important in determining the relations that exist among this subset of Klein links. With the ability to classify and categorize links based on their invariants, the following result was developed from patterns that were observed during the cataloging process. The following theorem discusses a reduction algorithm for determining the \( K(m,n) \) when \( m < n \). In most cases, the values of \( m \) and \( n \) will fall into the first case of the theorem; however, a second case is needed in order to address boundary conditions for these values.

**Theorem 3.** If \( m < n \), the \( K(m,n) \) Klein link is the disjoint union of \( K(m,m) \) and \( K(n-m,n-m) \).

**Proof. Case 1:** For \( n-m \geq 2 \) and \( 1 < m < n \), it is known from \[7\] that a Klein link can be written as

\[
K(m,n) = (\Gamma_{1,n-1})^m \prod_{i=1}^m (\Gamma_{n-1,i}^{-1}) \prod_{i=m+1}^{n-1} (\Gamma_{n-1,i}^{-1}).
\]

Expanding the first two components of \( K(m,n) \) yields:

\[
(\Gamma_{1,n-1})^m \prod_{i=1}^m (\Gamma_{n-1,i}^{-1}) = (\Gamma_{1,n-1})^m (\Gamma_{1,n-1}^{-1})\Gamma_{n-1,1}^{-1}\Gamma_{n-1,2}^{-1}\cdots\Gamma_{n-1,m}^{-1}.
\]

Looking at the middle two terms, we can see that the \( \Gamma_{1,n-1}^{-1} \) term and the \( \Gamma_{n-1,1}^{-1} \) term cancel completely. This results in

\[
(\Gamma_{1,n-1})^m (\Gamma_{1,n-1}^{-1})\Gamma_{n-1,1}^{-1}\Gamma_{n-1,2}^{-1}\cdots\Gamma_{n-1,m}^{-1} = (\Gamma_{1,n-1})^m (\Gamma_{1,n-1}^{-1})\Gamma_{n-1,1}^{-1}\Gamma_{n-1,2}^{-1}\cdots\Gamma_{n-1,m}^{-1}
\]

where there are \((m-1)\), \( \Gamma_{1,n-1}^{-1} \) terms remaining. Next, a \( \Gamma_{1,n-1}^{-1} \) term and the \( \Gamma_{n-1,2}^{-1} \) term cancel leaving a \( \Gamma_{1,1}^{-1} \) term, which results in

\[
(\Gamma_{1,n-1})^m (\Gamma_{1,n-1}^{-1})\Gamma_{n-1,1}^{-1}\Gamma_{n-1,2}^{-1}\cdots\Gamma_{n-1,m}^{-1} = (\Gamma_{1,n-1})^m (\Gamma_{1,n-1}^{-1})\Gamma_{n-1,1}^{-1}\Gamma_{n-1,2}^{-1}\cdots\Gamma_{n-1,m}^{-1} = (\Gamma_{1,n-1})^m (\Gamma_{1,n-1}^{-1})\Gamma_{n-1,1}^{-1}\Gamma_{n-1,2}^{-1}\cdots\Gamma_{n-1,m}^{-1}
\]

where there are \((m-2)\), \( \Gamma_{1,n-1}^{-1} \) terms remaining. Using Rule 3, the \( \Gamma_{1,1}^{-1} \) term can be moved from the middle of the above expression to the end as seen below,

\[
(\Gamma_{1,n-1})^m (\Gamma_{1,n-1}^{-1})\Gamma_{n-1,1}^{-1}\Gamma_{n-1,2}^{-1}\cdots\Gamma_{n-1,m}^{-1} = (\Gamma_{1,n-1})^m (\Gamma_{1,n-1}^{-1})\Gamma_{n-1,1}^{-1}\Gamma_{n-1,2}^{-1}\cdots\Gamma_{n-1,m}^{-1} = (\Gamma_{1,n-1})^m (\Gamma_{1,n-1}^{-1})\Gamma_{n-1,1}^{-1}\Gamma_{n-1,2}^{-1}\cdots\Gamma_{n-1,m}^{-1}
\]

Now, a \( \Gamma_{1,n-1}^{-1} \) term and the \( \Gamma_{n-1,3}^{-1} \) term cancel leaving a \( \Gamma_{1,2}^{-1} \) term. The expression can then be written as

\[
(\Gamma_{1,n-1})^m (\Gamma_{1,n-1}^{-1}) \Gamma_{1,2}^{-1}\Gamma_{n-1,1}^{-1}\Gamma_{n-1,2}^{-1}\cdots\Gamma_{n-1,m}^{-1} \Gamma_{1,1}^{-1}
\]

where there are \((m-3)\), \( \Gamma_{1,n-1}^{-1} \) terms remaining. Again, using Braid Rule 3, the \( \Gamma_{1,2}^{-1} \) term can be moved from the middle of the expression to the right,

\[
(\Gamma_{1,n-1})^m (\Gamma_{1,n-1}^{-1}) \Gamma_{1,2}^{-1}\Gamma_{n-1,1}^{-1}\Gamma_{n-1,2}^{-1}\cdots\Gamma_{n-1,m}^{-1} \Gamma_{1,2}^{-1}\Gamma_{1,1}^{-1}
\]
In this manner, the cancellation continues until it reaches a reduced form:

\[ \Gamma_{1,m-1}\Gamma_{1,m-2}\Gamma_{1,m-3} \cdots \Gamma_{1,3}\Gamma_{1,2}\Gamma_{1,1} = \prod_{i=1}^{m-1} \Gamma_{1,m-i}. \]

Thus, \( K(m,n) \) can now be written in the revised (disjoint) form:

\[ \prod_{i=1}^{m-1} \Gamma_{1,m-i} \prod_{i=m+1}^{n-1} \Gamma_{n-1,i}. \]

Similarly, let us expand the third component in the original expression of \( K(m,n) \).

\[ \prod_{i=m+1}^{n-1} \Gamma_{n-1,i}, \]

\[ = \Gamma_{n-1,m+1}^{-1}\Gamma_{n-1,m+2}^{-1} \cdots \Gamma_{n-1,n-2}^{-1}\Gamma_{n-1,n-1}^{-1}. \]

One can clearly see that this expression can also be written in the form:

\[ \prod_{i=1}^{n-m-1} \Gamma_{n-1,m+i}^{-1} \]

as this is the original notation with an adjustment on the indices. Thus, \( K(m,n) \) can now be written as:

\[ \prod_{i=1}^{m-1} \Gamma_{1,m-i} \prod_{i=m+1}^{n-1} \Gamma_{n-1,m+i}^{-1} \]

for the specific case when \( n - m \geq 2 \).

Now, let us consider \( K(m,m) \) which, following the general expression of \( K(m,n) \), can be written:

\[ K(m,m) = (\Gamma_{1,m-1})^{m} (\Gamma_{m-1,1})^{m} \Gamma_{m-1,2} \cdots \Gamma_{m-1,m-1} \]

\[ = (\Gamma_{1,1})^{m} (\Gamma_{m-1,1})^{m} \Gamma_{m-1,2} \cdots \Gamma_{m-1,m-1} \]

\[ = (\Gamma_{1,m-1})^{m-1} (\Gamma_{m-1,1})^{m-1} \Gamma_{m-1,2} \cdots \Gamma_{m-1,m-1} \]

\[ = (\Gamma_{1,m-1})^{m-2} (\Gamma_{1,1})^{m-2} (\Gamma_{1,2})^{m-3} \cdots \Gamma_{m-1,1} \]

\[ = (\Gamma_{1,m-1})^{m-3} (\Gamma_{1,1})^{m-3} (\Gamma_{1,2})^{m-4} \cdots \Gamma_{m-1,1} \]

\[ = (\Gamma_{1,m-1})^{m-4} (\Gamma_{1,1})^{m-4} (\Gamma_{1,2})^{m-5} \cdots \Gamma_{m-1,1} \]

\[ \vdots \]

\[ = \Gamma_{1,m-1}\Gamma_{1,m-2} \cdots \Gamma_{1,3}\Gamma_{1,2}\Gamma_{1,1} \]

\[ = \prod_{i=1}^{m-1} \Gamma_{1,m-i}. \]

This expression is equivalent to the reduced form of the first two components of Equation \[ \square \] thus, the first two components of \( K(m,n) \) are equivalent to \( K(m,m) \) when \( m < n \).

Next, let us consider \( K(n-m, n-m) \), using a similar method to the previous reduction,

\[ K(n-m, n-m) = \Gamma_{1,n-m-1}\Gamma_{1,n-m-2} \cdots \Gamma_{1,3}\Gamma_{1,2}\Gamma_{1,1}. \]

Writing the above expression in sigma notation results in:

\[ (\sigma_{1}\sigma_{2}\sigma_{3} \cdots \sigma_{n-m-1})(\sigma_{1}\sigma_{2}\sigma_{3} \cdots \sigma_{n-m-2}) \cdots (\sigma_{1}\sigma_{2}\sigma_{3})(\sigma_{1}\sigma_{2})\sigma_{1}. \]

Let us take the mirror image of this braid word for the \( K(n-m, n-m) \). This results in

\[ (\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{3}^{-1} \cdots \sigma_{n-m-1})(\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{3}^{-1} \cdots \sigma_{n-m-2}) \cdots (\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{3}^{-1})(\sigma_{1}^{-1}\sigma_{2}^{-1})\sigma_{1}^{-1}. \]

Claim: The above braid word is equivalent to \n
\[ \prod_{i=1}^{n-m-1} \Gamma_{n-1,m+i}^{-1} \]

Expanding the above form gives:

\[ \Gamma_{n-1,m+1}\Gamma_{n-1,m+2} \cdots \Gamma_{n-1,n-2}^{-1}\Gamma_{n-1,n-1}^{-1}. \]

Looking at this in sigma notation yields:

\[ (\sigma_{n-1}^{-1}\sigma_{n-2}^{-1} \cdots \sigma_{m-1}^{-1})(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1} \cdots \sigma_{m+2}^{-1}) \cdots (\sigma_{n-1}^{-1}\sigma_{n-2}^{-1})(\sigma_{n-1}^{-1}\sigma_{n-2}^{-1})\sigma_{n-1}^{-1}. \]

Appealing to Note \[ \square \] the braid represented by the above braid word can be reflected vertically which equals

\[ (\sigma_{m+1}^{-1}\sigma_{m+2}^{-1} \cdots \sigma_{m+3}^{-1})(\sigma_{m+1}^{-1}\sigma_{m+2}^{-1} \cdots \sigma_{m+2}^{-1}) \cdots (\sigma_{m+1}^{-1}\sigma_{m+2}^{-1})(\sigma_{m+1}^{-1}\sigma_{m+2}^{-1})\sigma_{n+2}^{-1}. \]

This braid word equals the original mirror image braid word for \( K(n-m, n-m) \) with a shift of indices. This shift in indices results because there is no strand interaction between the \( K(m,m) \) portion of the braid and the \( K(n-m, n-m) \) portion of the braid.
Thus, we have shown the $K(m,n)$ is the disjoint union of the $K(m,m)$ and the $K(n-m,n-m)^*$. 

**Case 2:** $n - m = 1$ or $m = 1$

In this case, the braid word of $K(n-m,n-m)$ cannot be obtained due to restrictions imposed by the formula in the creation of the braid word for a Klein link. Thus, this case when $n-m = 1$ can be proven with different methods from the first case. Alternatively, Figure 16 below serves as a sufficient proof that $K(m,n)$ is the disjoint union of $K(m,m)$ and $K(1,1)$ (which is equivalent to $K(n-m,n-m)$ in this case). Since the $K(1,1)$ is amphichiral (meaning it is equivalent to its mirror image), the mirror image in this $K(1,1)$ case is trivial. Note that the $K(1,1)$ portion of the resulting link is represented by the red component in Figure 16. As it crosses under every other strand in the braid, the red component can clearly be pulled under to the far right using a series of Reidemeister III moves. Thus, when the braid becomes closed, the red component forms the unknot and the remaining strands form $K(m,m)$.

![Figure 16: General braid representation of the $K(m,n)$ when $n - m = 1$.](image)

When $m = 1$, there are also restrictions imposed on the braid word that prevent it from being written for a Klein link in this case. Thus, again, Figure 17 will serve as a proof that the $K(m,n)$ is the disjoint union of the $K(1,1)$ and the $K(n-m,n-m)^*$. 

![Figure 17: General braid representation of the $K(m,n)$ when $n - m = 1$.](image)

The above braid word represents the same link as

$$
\prod_{i=1}^{4} \Gamma_{1,5-i} = \Gamma_{1,4}\Gamma_{1,3}\Gamma_{1,2}\Gamma_{1,1} = \sigma_{1}\sigma_{2}\sigma_{3}\sigma_{4}\sigma_{1}\sigma_{2}\sigma_{1}\sigma_{2}\sigma_{1}.
$$

Taking the mirror image of this braid word gives

$$
\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{3}^{-1}\sigma_{4}^{-1}\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{3}^{-1}\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{1}^{-1}.
$$

**Claim:** The above braid word represents the same link as

$$
\prod_{i=1}^{4} \Gamma_{6,2+i} = \prod_{i=1}^{6} \Gamma_{6,2+i} = \sigma_{6}^{-1}\sigma_{5}^{-1}\sigma_{4}^{-1}\sigma_{3}^{-1}\sigma_{6}^{-1}\sigma_{5}^{-1}\sigma_{4}^{-1}\sigma_{3}^{-1}\sigma_{6}^{-1}\sigma_{5}^{-1}\sigma_{4}^{-1}\sigma_{3}^{-1}.
$$

Appealing to Note 1, we can see that the vertical reflection of the braid associated to this braid word yields

$$
\sigma_{3}^{-1}\sigma_{4}^{-1}\sigma_{5}^{-1}\sigma_{6}^{-1}\sigma_{3}^{-1}\sigma_{4}^{-1}\sigma_{5}^{-1}\sigma_{3}^{-1}\sigma_{4}^{-1}\sigma_{5}^{-1}\sigma_{3}^{-1},
$$

which is the $K(5,5)^*$ shifted $m$ strands to the right as there is no interaction with the $K(2,2)$ portion of the braid.

7. **FUTURE WORK**

As noted, these results may be combined to extend the knowledge of our set of Klein links with $m < n$. A primary goal for future research will be developing formulas for determining invariants of Klein links where $m \geq n$. Also, we will investigate whether these formulas may be generalized for other sets of Klein links. Additional relations between Klein links and torus links will be investigated in hopes of finding new connections between existing links.

**Example 1.** Consider the $K(2,7)$. The $K(n-m, n-m)$ portion associated with this link is the $K(5,5)$. The braid word for the $K(5,5)$ can be expressed as

$$
\prod_{i=1}^{4} \Gamma_{1,5-i} = \Gamma_{1,4}\Gamma_{1,3}\Gamma_{1,2}\Gamma_{1,1} = \sigma_{1}\sigma_{2}\sigma_{3}\sigma_{4}\sigma_{1}\sigma_{2}\sigma_{1}.
$$

Looking at the mirror image of this braid word gives

$$
\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{3}^{-1}\sigma_{4}^{-1}\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{3}^{-1}\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{1}^{-1}.
$$

Taking the mirror image of this braid word gives

$$
\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{3}^{-1}\sigma_{4}^{-1}\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{3}^{-1}\sigma_{1}^{-1}\sigma_{2}^{-1}\sigma_{1}^{-1}.
$$

Claim: The above braid word represents the same link as

$$
\prod_{i=1}^{4} \Gamma_{6,2+i} = \prod_{i=1}^{6} \Gamma_{6,2+i} = \sigma_{6}^{-1}\sigma_{5}^{-1}\sigma_{4}^{-1}\sigma_{3}^{-1}\sigma_{6}^{-1}\sigma_{5}^{-1}\sigma_{4}^{-1}\sigma_{3}^{-1}\sigma_{6}^{-1}\sigma_{5}^{-1}\sigma_{4}^{-1}\sigma_{3}^{-1}.
$$

Appealing to Note 1, we can see that the vertical reflection of the braid associated to this braid word yields

$$
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$$

which is the $K(5,5)^*$ shifted $m$ strands to the right as there is no interaction with the $K(2,2)$ portion of the braid.

7. **FUTURE WORK**

As noted, these results may be combined to extend the knowledge of our set of Klein links with $m < n$. A primary goal for future research will be developing formulas for determining invariants of Klein links where $m \geq n$. Also, we will investigate whether these formulas may be generalized for other sets of Klein links. Additional relations between Klein links and torus links will be investigated in hopes of finding new connections between existing links.
8. REFERENCES


